Find points on sphere $x^2+y^2+z^2=4$ closest to and farthest from $(3,-2,1)$. 10/18

Soln: Optimize: Distance → Distance $(x,y,z),(3,-2,1)$

Subject to: Sphere → $x^2+y^2+z^2=4$

Equivalent: opt. $d^2 = (x-3)^2+(y+2)^2+(z-1)^2$

Subj. Sphere → $x^2+y^2+z^2=4$

try 4 lagrange multiplier here

Not necessary → improves QOL: $(x^2-6x+9)+(y^2+4y+4)+(z^2-2z+1)$ opt

$(x^2+y^2+z^2=4)$ subj.

opt: $(x^2+y^2+z^2)+(9+4+1)+(-6x+4y-2z)$

subj: $x^2+y^2+z^2=4$

opt: $18-6x+4y-2z$

subj: $x^2+y^2+z^2=4$

↪ $x^2+y^2+z^2-4=0$

opt: $f(x,y,z) = 18-6x+4y-2z$

Subj: $g(x,y,z)=0$ for $g(x,y,z)=x^2+y^2+z^2-4$

w/ $F(x,y,z,\lambda) = f(x,y,z)-\lambda g(x,y,z)$

$= 18-6x+4y-2z-\lambda(x^2+y^2+z^2-4)$

$\nabla \vec{F} = \vec{0}$

$\nabla F = \langle -6-2\lambda x, 4-2\lambda y, -2-2\lambda z, -(x^2+y^2+z^2-4)\rangle$ *by λ (1) → $\lambda \neq 0$

$$\therefore \nabla F = \vec{0} \text{ iff } \begin{cases} -6-2\lambda x = 0 \\ 4-2\lambda y = 0 \\ -2-2\lambda z = 0 \\ -(x^2+y^2+z^2-4)=0 \end{cases} \text{ iff } \begin{cases} \lambda x = -3 & (1) \\ \lambda y = 2 & (2) \\ \lambda z = -1 & (3) \\ x^2+y^2+z^2=4 & (4) \end{cases}$$

multiply (4) by λ^2.

$\lambda^2(x^2+y^2+z^2) = 4\lambda^2$ ie $(\lambda x)^2+(\lambda y)^2+(\lambda z)^2 = 4\lambda^2$

Now apply (1)(2)(3)

$(-3)^2+(2)^2+(-1)^2 = 4\lambda^2$

$14 = 4\lambda^2$ $\therefore \lambda = \pm\sqrt{\frac{7}{2}}$

Now remember $(1): \lambda x = -3$
$(2): \lambda y = 2$
$(3): \lambda z = -1$

w/ $\lambda = \pm\sqrt{\frac{7}{2}}$

2 Cases for λ:

<u>if $\lambda = \sqrt{\frac{7}{2}}$</u>: then solving (1)(2)(3) for x, y, z
yielding $\left(-3\sqrt{\frac{2}{7}}, 2\sqrt{\frac{2}{7}}, -\sqrt{\frac{2}{7}}\right) = A$

now $f(A) = 18 - 6\left(-3\sqrt{\frac{2}{7}}\right) + 4\left(2\sqrt{\frac{2}{7}}\right) - 2\left(-\frac{\sqrt{2}}{7}\right)$
$= 18 + 28\sqrt{\frac{2}{7}}$

<u>if $\lambda = -\sqrt{\frac{7}{2}}$</u>: then solving (1)(2)(3) for x, y, z
yielding $\left(3\sqrt{\frac{2}{7}}, -2\sqrt{\frac{2}{7}}, \sqrt{\frac{2}{7}}\right) = B$

now $f(B) = 18 - 6\left(3\sqrt{\frac{2}{7}}\right) + 4\left(-2\sqrt{\frac{2}{7}}\right) - 2\left(\sqrt{\frac{2}{7}}\right) = 18 - 28\sqrt{\frac{2}{7}}$

* global optimization comes from local optimization.

$\therefore$ $f(A) > f(B)$

$\therefore$ $f(A)$ is furtherest from $(3, -2, 1)$

$\therefore$ $f(b)$ is closest to $(3, -2, 1)$

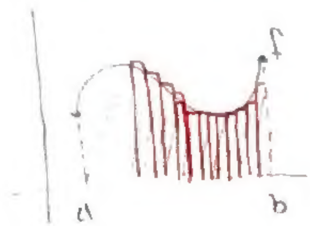
<u>Exercise</u>: Find the maximum volume of a box w/ no lid and surface area R. via lagrange multipliers.

Double Integral

Goal: integrate functions of 2 variable

↳ should an integral mean?

in Calc I: can integrate



$\int_a^b f(x)dx$ = "net area under graph of f"

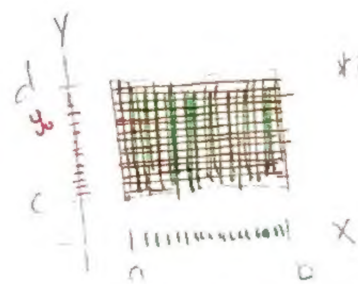
in Calc III: $\iint_R f(x,y)\,dA$

should represent the net volume under the graph of f above R.

- work w/ simplest possible regions: rectangles

$R = [a,b] \times [c,d]$

$= \{(x,y) \; x \in [a,b],\; y \in [c,d]\}$



*input space

R $[a,b] \times [c,d]$

*in Calc I,

to compute the definite integral $\int_a^b f(x)dx$, we "chunk" the interval $[a,b]$ and we approximated area via "left" endpoints adding rectangles area w/ height f(endpoints)

*in Calc III, $\iint_R f(x,y)\,dA$

is approx. by "chunking" rectangles and then picking some convention e.g. f(lower left endpoint)

for height. *limit the approx.

<u>Fubini's Theorem</u>: if $f(x,y)$ is cts on $R=[a,b]\times[c,d]$, then

$$\int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy = \iint_R f(x)\,dA = \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$$

* Start by fixing x ~~&~~ y first.

<u>NB</u>: hard $\ddot{n}$.

Ex: Compute $\iint_R x\sec^2(y)\,dA$ where $R=[1,3]\times[0,\frac{\pi}{4}]$

$$\int_0^{\pi/4}\int_1^3 x\sec^2(y)\,dx\,dy$$

$$\int_1^3\left[\frac{1}{2}x^2\sec(y)\right]_0^{\frac{\pi}{4}}dy \qquad \frac{1}{2}x^2\sec^2(y)\Big|_1^3 = 4\sec^2(y)$$

$$\int_1^3 4\sec^2(y)\,dy \rightarrow 4\tan(y)\Big|_1^3 = \boxed{4.}$$

<u>Other order</u>:

$$\int_1^3\int_0^{\pi/4} x\sec^2(y)\,dy\,dx \rightarrow \int_3^1 x\tan(y)\Big|_0^{\pi/4}dy$$

$$\int_3^1 x\,dx \rightarrow \boxed{4}$$

Ex: Compute $\iint_R \frac{1}{1+x+y}\,dA$ on $R=[1,2]\times[2,3]$

Sol. $$\int_2^3\int_1^2 \frac{1}{1+x+y}\,dx\,dy \rightarrow \Big[\ln|1+x+y|\Big]_1^2 \rightarrow \ln|1+2+y|-\ln|1+1+y|$$

$$\int_2^3 \ln|3+y|-\ln|2+y|\,dy \rightarrow 6\ln(6)-10\ln(5)+4\ln(4)-6+5+5-4$$

$$= 6\ln(6)-10\ln(5)+4\ln(4).$$